

Lecture 15—learning outcomes

- Recognise that photons have momentum and calculate this.
- Describe the Compton scattering experiment and calculate the Compton shift.
- Sketch the spectrum output from a blackbody radiator, explain the significance of Planck and Einstein's work in describing these curves.
- Use Wien's law to relate temperature and wavelength.
- Describe the relationships between deBroglie wavelength, momentum, speed and kinetic energy of particles.
- Use equations to describe the double-slit interference pattern observed for particles.
- Explain the different terms in the Schrödinger Equation.
- Understand how we can use the Schrödinger Equation in a simple situation to find the form of the wave function.
- Calculate the minimal uncertainty in a quantity using Heisenberg's uncertainty principle.

Textbook chapters: 38.3–7 <https://ebookcentral.proquest.com/lib/unsw/reader.action?c=UERG&docID=30765671&ppg=943>

Introduction to quantum mechanics

- The purpose of the lectures this week is to provide an introduction to quantum mechanics.
- In the early 1900's, physicists were grappling with the concept of wave-particle duality. When experimental evidence supporting the particle nature of light emerged, there were attempts to provide alternative explanations using waves, e.g., postulating that light consisted of wave packets.
- Even the photoelectric effect was open to wave interpretations.
- The evidence for particles became overwhelming with Compton's experiment in 1923, and then the explanation for the blackbody spectrum.
- It is now clear that light exhibits both wave-like and particle-like properties. Furthermore, “particles” such as electrons, protons, all the way up to macroscopic objects, share this wave-particle duality.
- Photons and electrons are quantum objects, with the most obvious difference being that photons are massless.

Feynman's Lectures on Physics, Volume III

For an excellent discussion on the fundamental ideas of quantum mechanics, see the first few chapters of this.

The Feynman Lectures on Physics, Volume III

QUANTUM MECHANICS

Feynman • Leighton • Sands

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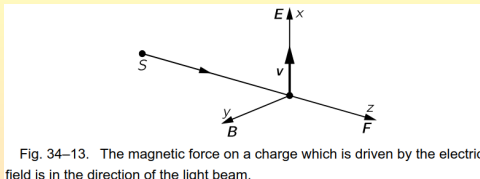
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https://www.feynmanlectures.caltech.edu/III_toc.html

Photons have momentum



https://www.feynmanlectures.caltech.edu/I_34.html

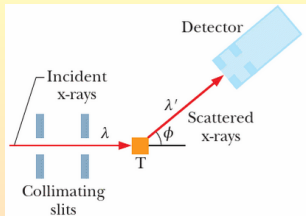
- Feynman has an interesting derivation (non-examinable) of the momentum of light, see the URL above. The basic idea is that if light hits a surface, it causes the charges (primarily electrons) to oscillate up and down. The moving charges then experience a force from the magnetic field component of the light. As you can see from the figure, this force $F = qvB$ is always along the direction of propagation of the light.
- Now, as we have seen, the E and B fields in light are related by $B = E/c$, so the time average of the magnetic force on the charges is $\langle F \rangle = q\langle vE \rangle/c$.
- Noting that qE is the electric force on the charge, then the instantaneous amount of work per second being done on the charge is $dW/dt = Fv = qev$. And therefore

$$\langle F \rangle = \frac{dW/dt}{c}$$

- Using the fact that $\langle F \rangle dt$ is the change in momentum, we have that $p = W/c$, so the momentum carried by light is $1/c$ times the energy carried.
- Finally, using Einstein's energy equation for a photon $W = hf$, we have that the momentum of a photon is

$$p = \frac{hf}{c} = \frac{h}{\lambda} \quad (\text{photon momentum})$$

Compton's experiment

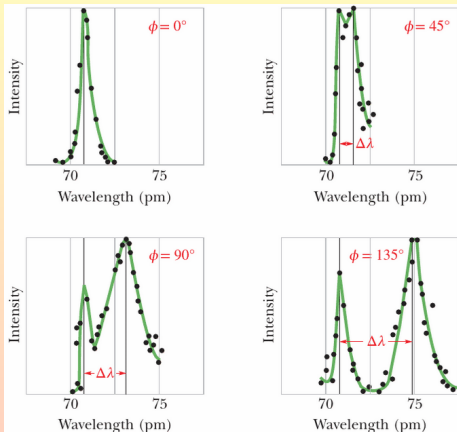


- In 1916, Einstein extended his concept of a photon with energy hf to have a momentum as derived in the previous slide

$$p = \frac{hf}{c} = \frac{h}{\lambda} \quad (\text{photon momentum})$$

- So, when a photon interacts with matter, it not only transfers energy, it transfers momentum (which is the source of radiation pressure).
- This was demonstrated experimentally by Compton in 1923 with the apparatus shown schematically in the figure. Incident x-rays with $\lambda = 71.1 \text{ pm}$ were directed onto a carbon target T . The scattered x-rays were observed at various angles θ , and their intensity and wavelength measured.
- Classically, the wavelength of the x-rays shouldn't change. But if light consists of photons, they would be able to exchange momentum with the electrons in the target, and thereby shift to a longer wavelength.
- Compton's experiment was a convincing demonstration of the particle nature of light, and he was awarded the Nobel Prize in 1927.

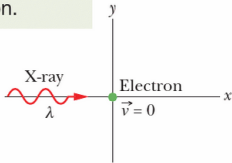
Compton's results



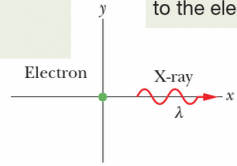
- The figure shows the results from Compton's experiment for four angles. At $\phi = 0^\circ$, with no change in direction of the x-rays, we see no wavelength shift..
- As the scattering angle ϕ increases, a second peak emerges at a wavelength $\Delta\lambda$ from the first, exactly as predicted by treating the interaction as a two particle collision between a photon and an electron.
- Interestingly, the first peak is still present, albeit at a reduced intensity. The first peak is due to x-rays interacting with electrons bound so tightly to the nucleus of the carbon atoms in the target that they can't be ejected from the atoms, and so the entire atom (with a mass of about 22,000 times the mass of an electron) is involved in the collision. The wavelength shift in this case is too small to be seen.

Compton scattering

An x-ray heads towards a target electron.



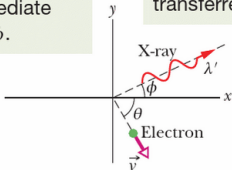
The x-ray can bypass the electron at scattering angle $\phi = 0$.



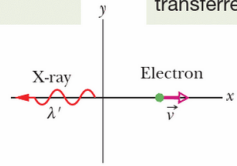
No energy is transferred to the electron.

Or it can scatter at some intermediate angle ϕ .

Intermediate energy is transferred.



Or it can backscatter at the maximum angle $\phi = 180^\circ$.



Maximum energy is transferred.

The figure shows an x-ray heading towards an isolated stationary electron. Three possible scattering angles are shown. The outcome can be predicted from simultaneously solving the momentum and energy conservation equations, much as a collision between two billiard balls, although we may need to take special relativity into account.

Derivation of the Compton scattering equation—part 1

We start with the conservation of energy, equating the energy hf of the incoming photon with the energy hf' of the scattered photon plus the kinetic energy K of the electron

$$hf = hf' + K$$

Since the electron may recoil with a speed comparable to that of light, we must use the relativistic expression for the kinetic energy

$$K = mc^2(\gamma - 1),$$

where m is the electron's mass and γ is the Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}}$$

Substituting for K in the conservation of energy equation yields

$$hf = hf' + mc^2(\gamma - 1)$$

Substituting c/λ for f and c/λ' for f' then leads to the new energy conservation equation

$$\frac{h}{\lambda} = \frac{h}{\lambda'} + mc(\gamma - 1)$$

Derivation of the Compton scattering equation–part 2

Next we apply the law of conservation of momentum to the x-ray-electron collision from two slides ago. The magnitude of the momentum of the incident photon is h/λ , and that of the scattered photon is h/λ' . The magnitude for the recoiling electron's momentum is $p = \gamma mv$. Because we have a two-dimensional situation, we write separate equations for the conservation of momentum along the x and y axes, obtaining

$$\frac{h}{\lambda} = \frac{h}{\lambda'} \cos \phi + \gamma mv \cos \theta \quad (x \text{ axis})$$

and

$$0 = \frac{h}{\lambda'} \sin \phi - \gamma mv \sin \theta \quad (y \text{ axis})$$

We want to find $\Delta\lambda (= \lambda' - \lambda)$, the Compton shift of the scattered x-rays. Of the five collision variables ($\lambda, \lambda', v, \phi$, and θ) that appear in our equations, we choose to eliminate v and θ , which deal only with the recoiling electron. Carrying out the algebra (it is somewhat complicated) leads to

$$\Delta\lambda = \frac{h}{mc} (1 - \cos \phi) \quad (\text{Compton shift})$$

This equation agrees exactly with Compton's experimental results. The quantity h/mc is a constant called the Compton wavelength. Its value depends on the mass m of the particle from which the x-rays scatter. Here that particle is a loosely bound electron, and thus we would substitute the mass of an electron for m to evaluate the Compton wavelength for Compton scattering from an electron.

Problem—Compton scattering of light by electrons

X-rays of wavelength
 $\lambda = 22 \text{ pm}$ (photon energy
 $h/\lambda = 56 \text{ keV}$) are
scattered from a carbon
target; the scattered rays
are detected at 85° to the
incident beam.

- (a) What is the Compton
shift of the scattered
rays?
- (b) What percentage of
the initial x-ray photon
energy is transferred
to an electron in such
scattering?

Solution

Compton scattering of light by electrons

X-rays of wavelength $\lambda = 22 \text{ pm}$ (photon energy = 56 keV) are scattered from a carbon target, and the scattered rays are detected at 85° to the incident beam.

(a) What is the Compton shift of the scattered rays?

KEY IDEA

The Compton shift is the wavelength change of the x-rays due to scattering from loosely bound electrons in a target. Further, that shift depends on the angle at which the scattered x-rays are detected, according to equation 38.11. The shift is zero for forward scattering at angle $\phi = 0^\circ$, and it is maximum for backscattering at angle $\phi = 180^\circ$. Here we have an intermediate situation at angle $\phi = 85^\circ$.

Calculations

Substituting 85° for that angle and $9.11 \times 10^{-31} \text{ kg}$ for the electron mass (because the scattering is from electrons) in equation 38.11 gives us

$$\begin{aligned}\Delta\lambda &= \frac{h}{mc} (1 - \cos \phi) \\ &= \frac{(6.63 \times 10^{-34} \text{ J} \cdot \text{s}) (1 - \cos 85^\circ)}{(9.11 \times 10^{-31} \text{ kg}) (3.00 \times 10^8 \text{ m/s})} \\ &= 2.21 \times 10^{-12} \text{ m} \approx 2.2 \text{ pm.} \quad (\text{Answer})\end{aligned}$$

(b) What percentage of the initial x-ray photon energy is transferred to an electron in such scattering?

KEY IDEA

We need to find the *fractional energy loss* (let us call it *frac*) for photons that scatter from the electrons:

$$\text{frac} = \frac{\text{energy loss}}{\text{initial energy}} = \frac{E - E'}{E}.$$

Calculations

From equation 38.2 ($E = hf$), we can substitute for the initial energy E and the detected energy E' of the x-rays in terms of frequencies. Then, from equation 38.1 ($f = c/\lambda$), we can substitute for those frequencies in terms of the wavelengths. We find

$$\begin{aligned}\text{frac} &= \frac{hf - hf'}{hf} = \frac{c/\lambda - c/\lambda'}{c/\lambda} = \frac{\lambda' - \lambda}{\lambda'} \\ &= \frac{\Delta\lambda}{\lambda + \Delta\lambda}.\end{aligned}$$

Substitution of data yields

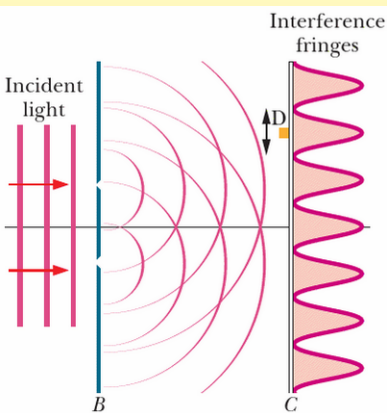
$$\text{frac} = \frac{2.21 \text{ pm}}{22 \text{ pm} + 2.21 \text{ pm}} = 0.091, \text{ or } 9.1\%. \quad (\text{Answer})$$

Although the Compton shift $\Delta\lambda$ is independent of the wavelength λ of the incident x-rays (see equation 38.11), our result here tells us that the *fractional* photon energy loss of the x-rays does depend on λ , increasing as the wavelength of the incident radiation decreases.

Why do only x-rays show Compton scattering? (Hint: this is not true)

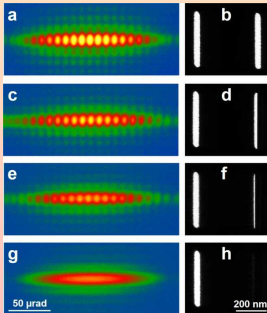
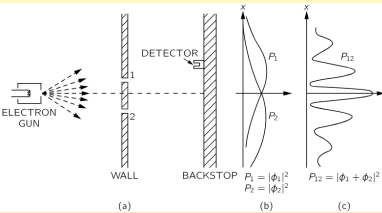
- In fact, Compton scattering occurs with photons of any wavelength interacting with any particles, it is just that the effect is more obvious with an x-ray hitting an isolated electron. Note that the x-ray has enough energy to separate an outer electron from its parent atom, effectively making an isolated electron.
- At visible wavelengths scattering from particles of size less than λ the Compton effect is negligible, and we use equations from Rayleigh scattering. When the particles become comparable to λ in size, we use Mie scattering.
- In astrophysics, we often see “inverse Compton scattering”, which is where a low energy photon (from, e.g., the cosmic microwave background) is scattered to higher energy by an electron that is more energetic than the photon.

Young's double-slit experiment revisited



- The figure shows the standard version of the double-slit experiment, but with the addition of a photon detector D at one point on screen C.
- If we move the detector up and down the screen, the photon detection rate varies with the intensity of the light, which in turn varies as the square of the amplitude of the electric field vector. This leads to the following realisation:
- *The probability (per unit time) that a photon will be detected in a small volume around a given point is proportional to the square of the amplitude of the electric field vector at that point.*

Young's double-slit experiment—the electron version

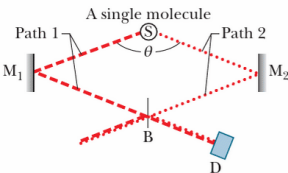


Tavabi et al. 2019, Scientific Reports, 9, 10458

- The double-slit experiment is nicely explained by the wave theory of light, at least where the quantisation of light is not obvious. Will interference occur with particles? We could try reducing the light intensity so that there is only one photon present in the experiment at any time, but let's be more ambitious and try it with electrons.
- The experiment has been done by, among others, Tavabi et al, 2019, with the results shown at bottom left. Electrons do indeed show interference effects between the two slits, whereas the distribution of electrons from a single slit only shows diffraction (panel (g)).
- If you consider electrons as point-like particles, it is hard to understand how an interference pattern can occur, even if electrons are sent through one at a time. How can an electron that passes through one slit be influenced by the existence of the other slit? It is almost as if an electron “sniffs out” the presence of both slits, and decides how it should move to make the observed fringes.
- The answer is that we have to abandon our classical concept of particles, and realise that electrons have wave-like properties that are understood through quantum mechanics.
- Light (photons) acts like wave and particles. And electrons act like waves and particles.

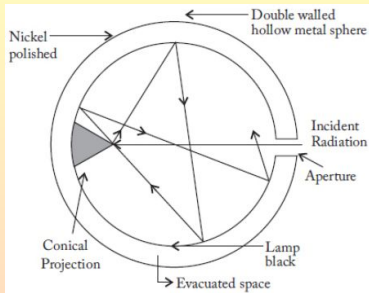
Young's double-slit experiment—the single-photon, wide-angle version

A *single* photon can take widely different paths and still interfere with itself.



- As a final example of a double-slit experiment, the figure at left shows one conducted by Lai and Diels in 1992.
- Their idea was to show that even when the possible paths of a photon are widely separated, it is still possible to show interference.
- The experiment was done without slits, and the interference was observed directly as a function of the phase difference between the two paths. The interference was observed even if only one photon was present in the experiment at any time.
- Once again, a particle (the photon) is best thought of as a probability wave that propagates from the source to the detector.
- Furthermore, any attempt to determine which path a photon has taken to the detector results in the interference pattern being destroyed. This is because the act of making the observation requires at the very least enough disturbance of the photon to move it by one fringe.

Blackbody radiation



■ The top figure shows a possible design for a **blackbody source**. The idea is that the inner spherical surface is kept at a constant temperature so that it emits photons due to thermal oscillations of the electrons. Any photon from outside that make their way in are “thermalised” by multiple interactions with the interior.

■ The bottom figure shows the observed spectrum of such a blackbody.

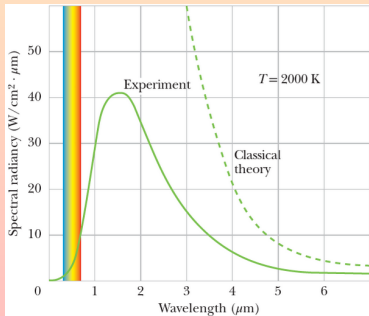
■ The prediction from classical theory is

$$S(\lambda) = \frac{2\pi ckT}{\lambda^4}$$

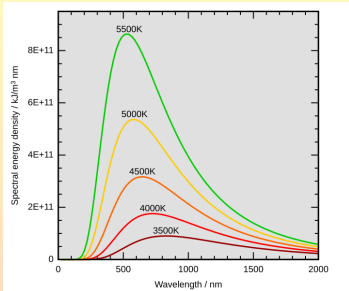
where $S(\lambda)$ is the **spectral radiance** in units of power, per unit area of the emitter, per unit wavelength, and $k = 1.38 \times 10^{-23}$ J/K is Boltzmann's constant. The classical result only agrees with experiment in the limit of long wavelengths.

■ In 1900 Planck found a formula that fitted the experimental results,

$$S(\lambda) = \frac{2\pi c^2 h}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} \quad (\text{Planck's radiation law})$$



Blackbody radiation—Einstein's contribution; $\lambda_{\max}$ and total power



- While Planck had come up with the correct equation for the blackbody spectrum, he did so using assumptions that weren't grounded in a physical model.
- In 1917 Einstein used his photon model to derive Planck's equation from two simple ideas: (1) the energies of the cavity-wall atoms emitting the radiation are quantised, and (2) the energies of the radiation in the cavity are quantised (into what we now call photons), each with energy $E = hf$.
- The wavelength $\lambda_{\max}$ at which the $S(\lambda)$ is maximum can be found by taking the first derivative of Planck's equation with respect to λ . The result is known as Wien's law:

$$\lambda_{\max} T = 2898 \mu\text{m} \cdot \text{K} \quad (\text{at maximum radiancy})$$

- If we integrate Planck's equation over all wavelengths, we find the power per unit area of a thermal radiator. If we then multiply by the total surface area A , we find the total radiated power P .

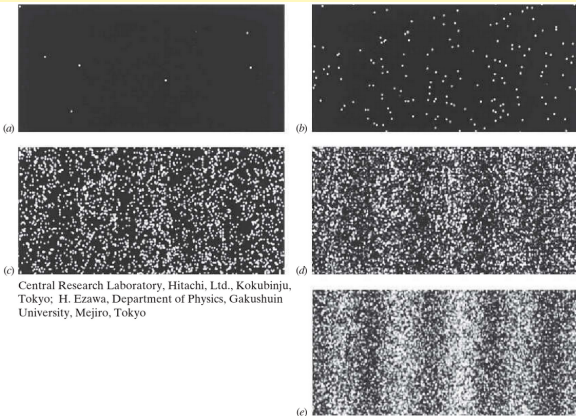
$$P = \sigma \varepsilon A T^4$$

where $\sigma = 5.6704 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4$ is the Stefan-Boltzmann constant and ε is the emissivity of the radiating surface ($\varepsilon = 1$ for an ideal blackbody radiator).



Inside a just-opened kiln

Electrons and de Broglie's "matter waves"



Central Research Laboratory, Hitachi, Ltd., Kokubinju, Tokyo; H. Ezawa, Department of Physics, Gakushuin University, Mejiro, Tokyo

- As we have seen, particles such as electrons exhibit wave-like properties. The figure shows the build up of interference fringes from electrons in a double-slit experiment.

- In 1924 de Broglie was the first to realise that particles of matter could have a wavelength. He associated a wavelength

$$\lambda = \frac{h}{p}$$

with a particle of momentum p . This wavelength is now known as the de Broglie wavelength, and was experimentally verified in 1927.

- These "matter waves" have now been seen in particles as large as iodine molecules and buckyballs (regular structures of carbon atoms, e.g., C_{60} and C_{70}).
- The non-relativistic kinetic energy E of a particle is given by $p^2/2m$. De Broglie's wavelength can be expressed as $\lambda = h/\sqrt{2mE}$. Also, $E = hf$.

Problem—de Broglie wavelength of an electron

What is the de Broglie wavelength of an electron with a kinetic energy of 120 eV?

Solution

De Broglie wavelength of an electron

What is the de Broglie wavelength of an electron with a kinetic energy of 120 eV?

KEY IDEAS

(1) We can find the electron's de Broglie wavelength λ from equation 38.17 ($\lambda = h/p$) if we first find the magnitude of its

momentum p . (2) We find p from the given kinetic energy K of the electron. That kinetic energy is much less than the rest energy of an electron (0.511 MeV, from table 37.3). Thus, we can get by with the classical approximations for momentum p ($= mv$) and kinetic energy ($= \frac{1}{2}mv^2$).

Calculations

We are given the value of the kinetic energy. So, in order to use the de Broglie relation, we first solve the kinetic energy equation for v and then substitute into the momentum equation, finding

$$\begin{aligned} p &= \sqrt{2mK} \\ &= \sqrt{(2) (9.11 \times 10^{-31} \text{ kg}) (120 \text{ eV}) (1.60 \times 10^{-19} \text{ J/eV})} \\ &= 5.19 \times 10^{-24} \text{ kg} \cdot \text{m/s}. \end{aligned}$$

From equation 38.17 then

$$\begin{aligned} \lambda &= \frac{h}{p} \\ &= \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{5.91 \times 10^{-24} \text{ kg} \cdot \text{m/s}} \\ &= 1.12 \times 10^{-10} \text{ m} = 112 \text{ pm}. \end{aligned} \quad (\text{Answer})$$

This wavelength associated with the electron is about the size of a typical atom. If we increase the electron's kinetic energy, the wavelength becomes even smaller.



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Schrödinger's equation for matter waves

- Following de Broglie's proposal that matter has an associated wavelength, and hence would show wave-like properties, Schrödinger realised that a wave equation was necessary to describe the probability of locating a particle in space and time.
- Recall the wave equation for photons in one dimension

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y(x, t)}{\partial t^2}$$

where v is the phase velocity of the light (c in a vacuum). The solutions to this are

$$y(x, t) = A \cos(kx - \omega t + \phi)$$

And the probability of finding a photon at (x, t) is proportional to y^2 (the light intensity).

- The fundamental difference between photons and matter is that matter has a non-zero mass, which has the consequence that its energy and momentum aren't fixed like a photon's and instead vary with the application of a force (or equivalently, travelling through regions of different potential energy $U(x, t)$). This makes the wave equation somewhat more complicated, and Schrödinger came up with the non-relativistic result

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} + U(x, t)\Psi(x, t) = i\hbar \frac{\partial \Psi(x, t)}{\partial t}$$

where $\Psi(x, t)$ is the wave function (Ψ is the Greek capital letter psi), $\hbar = h/2\pi$ (called “h-bar”, or sometimes “the reduced Planck's constant”), and $i^2 = -1$.

A superb video on Schrödinger's equation

The Schrödinger Equation

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} + V(x, t) \Psi(x, t) = i\hbar \frac{\partial \Psi(x, t)}{\partial t}$$

1) Specify the potential energy function $V(x, t)$

2) Solve the Schrödinger Equation to find $\Psi(x, t)$

But what does $\Psi(x, t)$ represent?

What is the Schrödinger Equation? A basic introduction to Quantum Mechanics



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<https://www.youtube.com/watch?v=2WPA1L9uJqo>



Schrödinger's equation for matter waves

- Let's examine a few properties of Schrödinger's equation

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} + U(x, t) \Psi(x, t) = i\hbar \frac{\partial \Psi(x, t)}{\partial t}$$

- Note that $\Psi(x, t)$ is a complex number (i.e., of the form $a + ib$, where $i^2 = -1$). This is a result of the equation involving both 1st and 2nd order derivatives of Ψ , leading to cos and sin terms that can't be equated without multiplying by i , unlike the wave equation for photons that only involves 2nd order derivatives, where a cos term remains a cos after differentiating twice.
- Unlike the wave equation for photons, where $y(x, t)$ can be interpreted as the E field, $\Psi(x, t)$ has no physical interpretation. However, in both cases the square of the wave functions (y^2 and $|\Psi|^2 = \Psi^* \Psi$, where Ψ^* is the complex conjugate of Ψ) are measurable quantities: they are proportional to the probability of finding a particle at the given location at the given time.
- Solving Schrödinger's equation is much easier if the potential U is independent of time, i.e., $U = U(x)$, which is often the case. This allows us to separate variables and write $\Psi(x, t) = \psi(x)\phi(t)$, which eliminates the partial derivatives and gives us two ordinary differential equations (ODEs).
- It is easy to show that the solution to the ODE involving time is $\phi(t) = e^{-iEt/\hbar}$ where E is the total energy of the particle. $\psi(x)$ is then given by the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + V(x) \psi(x) = E \psi(x)$$

Aside: eigenfunctions and eigenvalues

- When solving Schrödinger's equation you will often hear the terms **eigenfunction** and **eigenvalue**. These are defined as follows:
- *An eigenfunction is a function that, when operated on by an operator, results in a function that is proportional to the original function. The proportionality constant in this relationship is known as the eigenvalue.*
- In the case of Schrödinger's equation, the operator is d^2/dx^2 . And ψ is the eigenfunction.

Schrödinger's equation for matter waves

In this slide and the next, we will closely follow the textbook's discussion.

In 1926 Schrödinger came up with an equation for non-relativistic matter waves. By analogy with light waves, it has the property that *the probability of detecting a particle in a small volume centred on a given point in a matter wave is proportional to the value of $|\psi|^2$ at that point*. The value of ψ is a complex number, but $|\psi|^2$, the probability density, is always real.

Schrödinger's equation for a particle travelling in the x direction through a region in which it has a potential energy $U(x)$ is

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2} [E - U(x)]\psi = 0 \quad \begin{array}{l} \text{(Schrödinger's equation,} \\ \text{one-dimensional motion)} \end{array}$$

in which E is the total mechanical energy of the moving particle. In the special case of a constant U , independent of position and time, Schrödinger's equation can be rewritten as

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0 \quad \text{(Schrödinger's equation, constant } U),$$

where k is

$$k = \frac{2\pi\sqrt{2m(E - U)}}{h} \quad \text{(the angular wave number)}$$

The general solution is

$$\psi(x) = Ae^{ikx} + Be^{-ikx},$$

in which A and B are constants.

In the above we have ignored the time dependence of matter waves. In all cases we will meet, the space and time variables are separable, and we can write the full wave function $\Psi(x, y, z, t)$ as

$$\Psi(x, y, z, t) = \psi(x, y, z)e^{-i\omega t},$$

where $\omega (= 2\pi f)$ is the angular frequency of the matter wave.

Schrödinger's equation for matter waves

Continuing from the last slide, the full solution to Schrödinger's equation for a constant U is then

$$\begin{aligned}\Psi(x, t) &= \psi(x)e^{-i\omega t} = \left(Ae^{ikx} + Be^{-ikx}\right)e^{-i\omega t} \\ &= Ae^{i(kx - \omega t)} + Be^{-i(kx + \omega t)}\end{aligned}$$

which is the sum of two travelling waves, one moving in the $+x$ direction and one moving in $-x$. Note that we can switch between exponential and sinusoidal forms in this equation using Euler's formula

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \text{and} \quad e^{-i\theta} = \cos \theta - i \sin \theta$$

Let's evaluate the probability density $|\psi|^2$ for a particle with only positive motion. We eliminate the negative motion by setting B to zero, and then the solution at $t = 0$ becomes

$$\psi(x) = Ae^{ikx}$$

To calculate the probability density, we take the square of the absolute value:

$$|\psi|^2 = \left|Ae^{ikx}\right|^2 = A^2 \left|e^{ikx}\right|^2$$

Because

$$\left|e^{ikx}\right|^2 = \left(e^{ikx}\right) \left(e^{ikx}\right)^* = e^{ikx} e^{-ikx} = e^{ikx - ikx} = e^0 = 1,$$

we get

$$|\psi|^2 = A^2(1)^2 = A^2.$$

So, the end result is that the probability density is constant, and we have no idea where the particle can be! This is actually a consequence of Heisenberg's uncertainty principle that we will cover shortly. We have effectively exactly specified the momentum of the particle, which means that its position is entirely unknown.

Heisenberg's uncertainty principle: position and momentum

Heisenberg's uncertainty principle, proposed in 1927 by German physicist Werner Heisenberg states that measured values cannot be assigned to the position $\vec{r}$ and the momentum $\vec{p}$ of a particle simultaneously with unlimited precision.

In terms of $\hbar = h/2\pi$, the principle tells us

$$\Delta x \cdot \Delta p_x \geq \hbar$$

$$\Delta y \cdot \Delta p_y \geq \hbar \quad (\text{Heisenberg's uncertainty principle}).$$

$$\Delta z \cdot \Delta p_z \geq \hbar$$

Here Δx and Δp_x represent the intrinsic uncertainties in the measurements of the x components of $\vec{r}$ and $\vec{p}$, with parallel meanings for the y and z terms. Even with the best measuring instruments, each product of a position uncertainty and a momentum uncertainty will be greater than $\hbar$, never less.

This principle can be understood as a result of using a wave equation to represent the position/momentum of a particle: to locate the particle more precisely it is necessary to compress the wave packet into a smaller space, but this necessarily leads to a reduced ability to measure the wavelength (and hence momentum) of the particle since the number of waves in the packet has an uncertainty of ~ 1 . To know the wavelength more accurately, we need to count more wavelengths, so we need a longer wave packet, and therefore lose position information.

Veritasium on Heisenberg's uncertainty principle



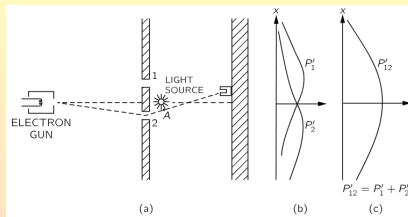
Heisenberg's Uncertainty Principle Explained

Veritasium 19.4M subscribers

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<https://www.youtube.com/watch?v=a8FTr2qMutA>

Feynman on Heisenberg's uncertainty principle and the double-slit experiment



- The figure shows the double-slit experiment for electrons. P'_1 is the probability distribution on the screen when slit 2 is closed. P'_2 is when slit 1 is closed. (Interference isn't shown).
- Suppose we want to determine for each electron which slit they go through. We can do this with a bright light source behind the slits that bounces light off the electrons into a microscope focused on the slits.
- When you do this, however, the interference pattern from the two slits disappears, as shown by P'_{12} .
- The photons from the light source cause a change in momentum of the electrons, which is sufficient to scramble the interference.
- So, let's reduce the light intensity so that only a single photon at most hits the electrons. The interference pattern begins to reappear, but it is only composed of those electrons that weren't hit by any photons!
- How about using lower energy photons? When we do this, the interference pattern reappears, but the wavelength of the photons is now longer than the slit separation, and we cannot determine which slit the electron went through.
- The lesson here is that no matter how hard you try, you can't beat Heisenberg's uncertainty principle!

Problem—Uncertainty principle: position and momentum

Assume that an electron is moving along an x axis and that you measure its speed to be 2.05×10^6 m/s, which can be known with a precision of 0.50%. What is the minimum uncertainty (as allowed by the uncertainty principle in quantum theory) with which you can simultaneously measure the position of the electron along the x axis?

Solution

Uncertainty principle: position and momentum

Assume that an electron is moving along an x axis and that you measure its speed to be 2.05×10^6 m/s, which can be known with a precision of 0.50%. What is the minimum uncertainty (as allowed by the uncertainty principle in quantum theory) with which you can simultaneously measure the position of the electron along the x axis?

KEY IDEA

The minimum uncertainty allowed by quantum theory is given by Heisenberg's uncertainty principle in equation 38.28. We need only consider components along the x axis because we have motion only along that axis and want the uncertainty Δx in location along that axis. Since we want the minimum allowed uncertainty, we use the equality instead of the inequality in the x -axis part of equation 38.28, writing $\Delta x \cdot \Delta p_x = \hbar$.

Calculations

To evaluate the uncertainty Δp_x in the momentum, we must first evaluate the momentum component p_x . Because the electron's speed v_x is much less than the speed of light c ,

we can evaluate p_x with the classical expression for momentum instead of using a relativistic expression. We find

$$\begin{aligned} p_x &= mv_x = (9.11 \times 10^{-31} \text{ kg}) (2.05 \times 10^6 \text{ m/s}) \\ &= 1.87 \times 10^{-24} \text{ kg} \cdot \text{m/s}. \end{aligned}$$

The uncertainty in the speed is given as 0.50% of the measured speed. Because p_x depends directly on speed, the uncertainty Δp_x in the momentum must be 0.50% of the momentum:

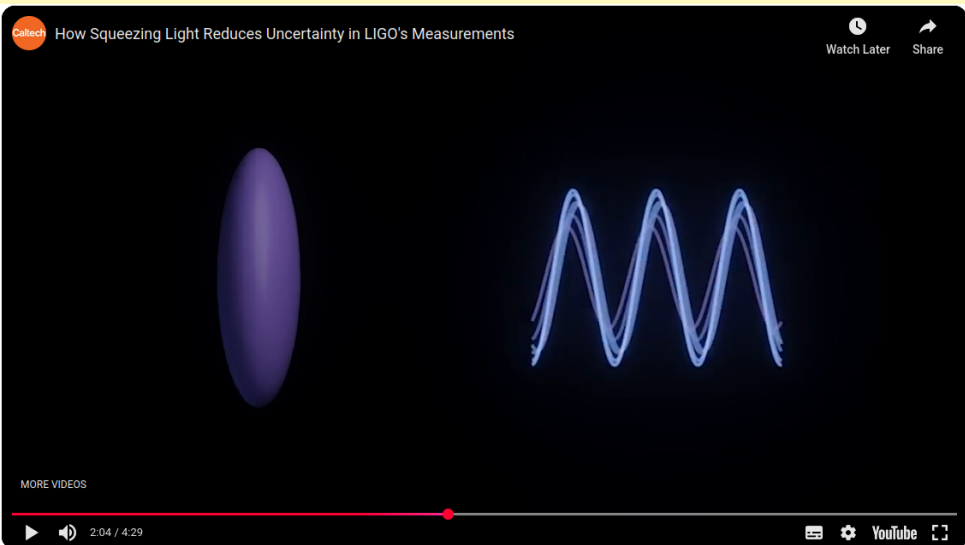
$$\begin{aligned} \Delta p_x &= (0.0050) p_x \\ &= (0.0050) (1.87 \times 10^{-24} \text{ kg} \cdot \text{m/s}) \\ &= 9.35 \times 10^{-27} \text{ kg} \cdot \text{m/s}. \end{aligned}$$

Then the uncertainty principle gives us

$$\begin{aligned} \Delta x &= \frac{\hbar}{\Delta p_x} = \frac{(6.63 \times 10^{-34} \text{ J} \cdot \text{s}) / 2\pi}{9.35 \times 10^{-27} \text{ kg} \cdot \text{m/s}} \\ &= 1.13 \times 10^{-8} \text{ m} \approx 11 \text{ nm}, \end{aligned} \quad (\text{Answer})$$

which is about 100 atomic diameters.

"Squeezed light" manipulates Heisenberg's uncertainty principle to double LIGO's sensitivity



<https://www.youtube.com/watch?v=q5lnamXhnfI>